11. Markeev, A. P., On the stability of the Lagrange solutions of a spatial elliptic three-body problem. Celestial Mechanics, Vol. 8, №3, 1973.

Translated by N. H. C.

UDC 531.36

CERTAIN STABILITY QUESTIONS IN THE PRESENCE OF RESONANCES

PMM Vol. 38, №1, 1974, pp. 56-65
G. G. KHAZINA
(Moscow)
(Received October 9, 1972)

We study questions of the stability of the equilibrium position of nonlinear systems neutral in the linear approximation. We obtain necessary and sufficient stability conditions in the presence of one resonance, as well as some results concerning the interaction of several resonances. We show that Liapunov instability follows from instability in finite order.

1. We consider a system of ordinary differential equations with real coefficients

$$dx_\alpha / dt = A_\alpha^\beta x_\beta + A_\alpha^{\beta\gamma} x_\beta x_\gamma + \ldots, \qquad \alpha = 1, \ldots, n \tag{1.1}$$

We study the stability of the equilibrium position $x_1 = \ldots = x_n = 0$ (relative to variations of the initial data) if the eigenvalues of the linearized system are purely imaginary, simple, and nonzero (Condition (A)) Under these conditions the question of the stability of the equilibrium position in the resonance-free case was examined by Molchanov (*). This question has been studied for Hamiltonian systems in the presence of resonances of arbitrary order [1]. The case of one third-order resonance was considered in [2] for general systems. In the present paper we have obtained necessary and sufficient conditions for the stability of the equilibrium position of system (1.1) in second order by perturbation theory in the presence of parametric resonance . We have proved the Liapunov-instability of the equilibrium position of system (1.1) in the presence of an arbitrary third-order resonance if the system is Birkhoff-unstable (in second order) and we have examined the question of the interaction of two or of several resonances. In particular, we have shown that the interaction of two resonances can lead to instability even when each resonance individually does not cause instability.

Let $\lambda_1, \ldots, \lambda_l, \ -\lambda_1, \ldots, -\lambda_l$ be the eigenvalues (frequencies) of the system being analyzed $(2l = n)$. We say that system (1.1) possesses k th-order resonance if integers k_m $(m = 1, \ldots, l)$, exist, not all equal to zero, $|k_1| + \ldots + |k_l| = k$, such that $k_1\lambda_1 + \ldots + k_l\lambda_l = 0$. (For example, relations of the form

$$\lambda_i - 2\lambda_j = 0, \ \lambda_i + \lambda_j + \lambda_k = 0, \ \lambda_i + \lambda_j - \lambda_k = 0$$

exhaust all third-order resonances). The vector $(k_1, \ldots, k_l)$ is said to be resonant.

---

*) Molchanov, A. M., On the stability of nonlinear systems. Thesis for a Doctor's degree, Moscow, 1962.

When Condition (A) is satisfied, system (1.1) can be reduced by a quadratic change of variables to the normal form (the asterisk denotes conjugation)

$$dy_\alpha / dt = \lambda_\alpha y_\alpha + B_\alpha^{\beta\gamma} y_\beta y_\gamma + \ldots \tag{1.2}$$
$$dy_\alpha^* / dt = \lambda_\alpha^* y_\alpha^* + (B_\alpha^{\beta\gamma})^* y_\beta^* y_\gamma^* + \ldots$$
$$\alpha = 1, \ldots, l; \quad \beta, \gamma = 1, \ldots, l, 1^*, \ldots, l^*; \quad y_{\alpha^*} = y_\alpha^*$$

The system

$$dy_\alpha/dt = \lambda_\alpha y_\alpha + B_\alpha^{\beta\gamma} y_\beta y_\gamma \tag{1.3}$$
$$dy_\alpha^*/dt = \lambda_\alpha^* y_\alpha^* + (B_\alpha^{\beta\gamma})^* y_\beta^* y_\gamma^*$$
$$\alpha = 1, \ldots, l; \quad \beta, \gamma = 1, \ldots, l, 1^*, \ldots, l^*; \quad y_{\alpha^*} = y_\alpha^*$$

obtained from (1.2) by discarding all terms higher than second order, is called truncated system and we say that (1.2) is stable (unstable) in second order if its truncated form (1.3) is stable (unstable).

Let system (1.1) possess the (parametric) resonance $\lambda_2 - 2\lambda_1 = 0$. The first group of Eqs. (1.3) then has the following form (the equations for the conjugate quantities are computed analogously):

$$dy_1 / dt = \lambda_1 y_1 + B_1^{21*} y_2 y_1^*, \quad dy_2 / dt = \lambda_2 y_2 + B_2^{11} y_1^2 \tag{1.4}$$
$$dy_\alpha / dt = \lambda_\alpha y_\alpha, \quad \alpha = 3, \ldots, l$$

Passing to a polar coordinate system, $y_\alpha = \rho_\alpha e^{i\varphi_\alpha}$, $\alpha = 1, \ldots, l$, we obtain

$$\frac{d\rho_j^2}{dt} = 2\rho_1^2\rho_2 P_j(\psi), \quad \frac{d\psi}{dt} = 2\rho_1^2\rho_2\left(\frac{P_1'}{\rho_1^2} + \frac{P_2'}{2\rho_2^2}\right), \quad j = 1,2 \tag{1.5}$$
$$\frac{d\rho_\alpha^2}{dt} = 0, \quad \frac{d\varphi_\alpha}{dt} = \frac{\lambda_\alpha}{i} \quad \alpha = 1, \ldots, l$$

where

$$\psi = \varphi_2 - 2\varphi_1, \ P_j = A_j \cos\psi + B_j \sin\psi, \ P_j' = dP_j / dt, \quad j = 1, 2$$
$$A_1 = \operatorname{Re} B_1^{21*}, \quad B_1 = \operatorname{Im} B_1^{21*}, \quad A_2 = \operatorname{Re} B_2^{11},$$
$$B_2 = -\operatorname{Im} B_2^{11}$$

Theorem 1. The equilibrium position $(\rho_1 = \ldots = \rho_l = 0)$ of system (1.5) is stable if and only if the condition

$$A_1 = -\gamma A_2, \quad B_1 = -\gamma B_2, \quad \gamma > 0 \tag{1.6}$$

are satisfied.

Proof. If conditions (1.6) are fulfilled, system (1.5) has the integral $I = \rho_1^2 + \gamma\rho_2^2 + \rho_3^2 + \ldots + \rho_l^2$, whose existence guarantees stability. Now suppose that conditions (1.6) are not fulfilled. Let us show that then system (1.5) has a growing solution of the type of an invariant ray

$$\rho_\alpha(t) = k_\alpha b(t), \ k_\alpha > 0, \quad b^{\cdot} > 0, \quad b(0) > 0, \quad \alpha = 1, 2, \tag{1.7}$$
$$\psi = \psi_0 = \text{const}$$

Substituting (1.7) into (1.5), we obtain

$$b^{\cdot} = k_2 P_1(\psi_0)\, b^2, \qquad b^{\cdot} = \frac{k_1^2}{k_2} P_2(\psi_0)\, b^2 \tag{1.8}$$

$$\psi^{\cdot} = 2k_1^2 k_2 \left(\frac{P_1'(\psi_0)}{k_1^2} + \frac{P_2'(\psi_0)}{2k_2^2}\right) b$$

A solution of form (1.7) of system (1.5) exists if we can find $k_1 > 0$, $k_2 > 0$, such that

$$(B_1 k_2^2 - B_2 k_1^2)\sin\psi_0 + (A_1 k_2^2 - A_2 k_1^2)\cos\psi_0 = 0 \tag{1.9}$$

$$(2A_1 k_2^2 + A_2 k_1^2)\sin\psi_0 - (2B_1 k_2^2 + B_2 k_1^2)\cos\psi_0 = 0$$

$$(P_1(\psi_0) > 0)$$

The first relation of (1.9) is obtained by equating the right-hand sides of the first equalities in (1.8); the second relation of (1.9) is obtained from the vanishing of the right-hand side of the last relation in (1.8). The inequality within parentheses can be satisfied by taking $\psi_0 + \pi$ instead of ψ_0.

System (1.9) as a system of linear equations in $\sin\psi_0$ and $\cos\psi_0$ is consistent if its determinant equals zero, i.e.

$$2(A_1^2 + B_1^2)\varkappa^2 - (A_1 A_2 + B_1 B_2)\varkappa - (A_2^2 + B_2^2) = 0, \quad \varkappa = k_2^2 / k_1^2$$

This equation in $\varkappa$ has a positive root $\varkappa_0$. We see that when $\varkappa = \varkappa_0$ we can find ψ_0 from (1.9) such that $P_1(\psi_0) > 0$. (We note that when (1.6) are satisfied, a positive root $\varkappa_0 = 1 / 2\gamma$ exists as well, but for this $\varkappa_0$ the first equality of (1.9) turns into $P_1(\psi_0) = 0$, so that the condition within parentheses in (1.9) is not fulfilled). Thus, a solution of form (1.7) of system (1.5) exists, and $db / dt = n^2 b^2$, $n \neq 0$, whence instability follows. The theorem is proved.

We say that a resonance is included if the corresponding coefficient $B_\alpha^{\beta\gamma}$ of the resonance term is not equal zero. A resonance is said to be essential or unessential depending on whether it leads to instability or not with the rest of resonances excluded. In these terms Theorem 1 can be formulated in the following manner: the resonance $\lambda_2 - 2\lambda_1 = 0$ is essential if and only if one of the solutions of system (1.5) is an invariant ray.

An analogous assertion is true for the resonant vectors $k(1, -1, -1, 0, \ldots, 0)$, $k(1, 1, 1, 0, \ldots, 0)$ (see [2]). Third-order resonances can only be of the types indicated; therefore, the following general statement is valid.

Theorem 2. Let system (1.1) possess one (arbitrary) resonance of third order. For the resonance to be essential in the second order it is necessary and sufficient that among the solutions of the truncated system there be a growing solution of the invariant ray type.

We note that the resonance 1 : 2 is almost always essential, whereas the resonance 1 : 1 : 1 leads to instability in only half the cases.

2. Theorem 3. If system (1.1) possesses the resonance $\lambda_2 - 2\lambda_1 = 0$, then Liapunov-instability follows from instability in second order of the equilibruim position.

The presence among the solutions of the analog system of a specific solution (an invariant ray) is a necessary and sufficient condition for the instability of the truncated system. The complete system (1.1), differing from the truncated one only by higher terms, may not have such a solution. However, it turns out that the complete system's solutions in some neighborhood of the invariant ray of the truncated system remain growing.

Proof. Under the conditions indicated system (1.2) appears in the following form (the equations for the conjugate quantities are computed analogously):

$$x_1^{\bullet} = \lambda_1 x_1 + B_1^{1*2} x_1^* x_2 + x_1 B_1^{jj^*} x_j x_j^* + R_1 \tag{2.1}$$
$$x_2^{\bullet} = \lambda_2 x_2 + B_2^{11} x_1^2 + x_2 B_2^{jj*} x_j x_j^* + R_2$$
$$x_k^{\bullet} = \lambda_k x_k + x_k B_k^{jj*} x_j x_j^* + R_k, \quad k = 3, \ldots, l$$

Here R_k denotes the higher-order terms; the degree of R_1, R_2 in the variables $x_1, \ldots, x_l^*$ is higher than three, the degree of $R_3, \ldots, R_l$ is higher than four. In the variables ρ_α, φ_α ($x_\alpha = \rho_\alpha e^{i\varphi_\alpha}$, $\alpha = 1, \ldots, l$) we write down only that subsystem which does not contain $\varphi_2^{\bullet}, \ldots, \varphi_l^{\bullet}$

$$\frac{d\rho_\alpha^2}{dt} = 2\rho_1^2\rho_2 P_\alpha(\overline{\psi}) + \rho_\alpha^2 (C_1^\alpha \rho_1^2 + C_2^\alpha \rho_2^2 + S_\alpha) + \bar{R}_\alpha, \quad \alpha = 1, 2 \tag{2.2}$$
$$\frac{d\rho_\beta^2}{dt} = \rho_\beta^2 (C_1^\beta \rho_1^2 + C_2^\beta \rho_2^2 + S_\beta) + \bar{R}_\beta, \quad \beta = 3, \ldots, l$$
$$\frac{d\overline{\psi}}{dt} = 2\rho_1^2\rho_2 \left(\frac{P_1'}{\rho_1^2} + \frac{P_2'}{2\rho_2^2}\right) + (L_1\rho_1^2 + L_2\rho_2^2 + N) + \bar{R}_{l+1}$$
$$P_j(\overline{\psi}) = A_j \cos\overline{\psi} + B_j \sin\overline{\psi}, \ \overline{\psi} = \varphi_2 - 2\varphi_1, \ \bar{R}_\alpha = \bar{R}_\alpha(\rho_1, \ldots, \rho_l, \varphi_j \overline{\psi})$$
$$N = \sum_{j=3}^{l} L_j \rho_j^2, \quad P_j' = \frac{dP}{d\overline{\psi}}, \quad S_\alpha = \sum_{j=3}^{l} C_j^\alpha \rho_j^2$$

Here A_j, B_j, C_i^j, L_i are real coefficients, while $\bar{R}_\alpha$ denotes terms of higher orders in comparison with the written ones.

The conditions for the existence of an invariant ray in the truncated system are the following (see (1.9)):

$$P_2(\overline{\psi}_0) = k^2 P_1(\overline{\psi}_0), \ P_2'(\psi_0) = -2k^2 P_1'(\overline{\psi}_0), \ P_1(\overline{\psi}_0) > 0 \ (k = k_2 / k_1) \tag{2.3}$$

Using these conditions we reduce system (2.2) to a more convenient form. At first we introduce the variables r, $\overline{\varphi}$: $\rho_1 = k^{-1} r \sin\overline{\varphi}$, $\rho_2 = r\cos\overline{\varphi}$. In the variables r, $\overline{\varphi}$, $\rho_3, \ldots, \rho_l$ system (2.2) takes the form

$$\frac{dr}{dt} = r^2\left(P_1 + \frac{P_2}{k^2}\right)\sin^2\overline{\varphi}\cos\overline{\varphi} + \frac{r}{2}(S_1 \sin^2\overline{\varphi} + S_2\cos^2\overline{\varphi}) + \tag{2.4}$$
$$\frac{r^3}{2k^2}\sin^2\overline{\varphi}\left(\frac{C_1^1}{k^2}\sin^2\overline{\varphi} + C_1^2\cos^2\overline{\varphi}\right) +$$
$$\frac{r^3}{2}\cos^2\overline{\varphi}\left(\frac{C_2^1}{k^2}\sin^2\overline{\varphi} + C_2^2\cos^2\overline{\varphi}\right) + \bar{R}_0^1$$
$$\frac{d\overline{\varphi}}{dt} = r\sin\overline{\varphi}\left(P_1\cos^2\overline{\varphi} - \frac{P_2}{k^2}\sin^2\overline{\varphi}\right) +$$
$$\frac{r^2}{2}\sin\overline{\varphi}\cos\overline{\varphi}\left(\frac{M_1}{k^2}\sin^2\overline{\varphi} + M_2\cos^2\overline{\varphi}\right) + M + \bar{R}_1^1$$
$$\frac{d\overline{\psi}}{dt} = \frac{r}{\cos\overline{\varphi}}\left(2P_1'\cos^2\overline{\varphi} + \frac{P_2'}{k^2}\sin^2\overline{\varphi}\right) +$$
$$r^2\left(\frac{L_1}{k^2}\sin^2\overline{\varphi} + L_2\cos^2\overline{\varphi}\right) + N + \bar{R}_2^1$$
$$\frac{d\rho_\alpha^2}{dt} = \rho_\alpha^2\left(\frac{C_1^\alpha}{k^2}\sin^2\overline{\varphi} + C_2^\alpha\cos^2\overline{\varphi}\right) + S_\alpha\rho_\alpha^2 + \bar{R}_\alpha^1, \quad \alpha = 3, \ldots, l$$

$$M_j = C_j^1 - C_j^2, \qquad M = \sum_{j=3}^{l} M_j \rho_j^2$$

Here the degree of $\bar{R}_0^1$ in $r, \rho_3, \ldots, \rho_l$ is higher than four, the degree of $\bar{R}_1^1$, $\bar{R}_2^1$ is higher than two, and the degree of $\bar{R}_3^1, \ldots, \bar{R}_l^1$ is higher than five. The values $\bar{\varphi} = \pi / 4, \bar{\psi} = \bar{\psi}_0$ correspond to the truncated system's invariant ray. Making the substitution $\varphi = \bar{\varphi} - \pi / 4, \psi = \bar{\psi} - \bar{\psi}_0$ and expanding the right-hand sides in a Taylor series in a neighborhood of $\varphi = 0$, $\psi = 0$, by taking the existence conditions for ray (2.3) into account and restricting ourselves only to terms of first order in φ and ψ, we finally write the system as

$$\frac{dr}{dt} = \frac{r^2}{2\sqrt{2}}(2P_1^\circ + 2P_1^\circ\varphi - P_1^{\circ\prime}\psi) + \frac{r}{4}(S_1 + S_2) + Q_0 \qquad (2.5)$$

$$\frac{d\varphi}{dt} = \frac{r}{2\sqrt{2}}(-4P_1^\circ\varphi + 3P_1^{\circ\prime}\psi) + Er^2 + \frac{1}{4}M + Q_1$$

$$\frac{d\psi}{dt} = \frac{r}{\sqrt{2}}(-8P_1^{\circ\prime}\varphi - 3P_1^\circ\psi) + \frac{r^2}{2}\left(\frac{L_1}{k^2} + L_2\right) + N + Q_2$$

$$\frac{d\rho_\alpha^2}{dt} = \frac{r^2}{2}\rho_\alpha^2\left(\frac{C_1^\alpha}{k^2} + C_2^\alpha\right) + S_\alpha\rho_\alpha^2 + Q_\alpha, \quad \alpha = 3, \ldots, l$$

$$P_\alpha^\circ = P_\alpha(\bar{\psi}_0), \qquad P_\alpha^{\circ\prime} = P_\alpha'(\bar{\psi}_0), \qquad E = \frac{1}{8}\left(\frac{M_1}{k^2} + M_2\right)$$

Here Q_α denotes higher-order terms.

We now show that for a suitable choice of δ the function

$$F(r, \varphi, \psi, \rho_3, \ldots, \rho_l) = \varphi^2 + \delta^2\psi^2 + \rho_3 + \ldots + \rho_l - r$$

is a Chetaev function for system (2.5), i.e. in the region $F \leqslant 0$, by virtue of (2.5), the derivative $dF / dt < 0$. We have

$$\frac{dF}{dt} = 2\varphi\left\{\frac{r}{2\sqrt{2}}(-4P_1^\circ\varphi + 3P_1^{\circ\prime}\psi) + \left[Er^2 + \frac{M}{4} + Q_1\right]\right\} + \qquad (2.6)$$

$$2\delta^2\psi\left\{\frac{r}{\sqrt{2}}(-8P_1^{\circ\prime}\varphi - 3P_1^\circ\psi) + \left[\frac{r^2}{2}\left(\frac{L_1}{2} + L_2\right) + N + Q_2\right]\right\} +$$

$$\sum_{j=3}^{l}\frac{r^2}{4}\rho_j\left(\frac{C_1^j}{k^2} + C_2^j\right) + \left[\frac{S_j\rho_j}{2} + Q_j\right] -$$

$$\left\{\frac{r^2}{\sqrt{2}}P_1^\circ + \left[\frac{r^2}{2\sqrt{2}}(2P_1^\circ\varphi - P_1^{\circ\prime}\psi) + \frac{r}{4}(S_1 + S_2) + Q_0\right]\right\}$$

Because

$$\varphi^2 \leqslant r, \quad \delta^2\psi^2 \leqslant r, \quad \rho_3 + \rho_4 + \ldots + \rho_l \leqslant r \quad (\rho_j \geqslant 0)$$

in the region being considered, the terms within the brackets are unessential in comparison with quantities of the order of r^2 at sufficiently small r.

Let us show that we can choose δ such that the expression

$$-r^2P_1^0 - 16\delta^2 r\varphi\psi P_1^{\circ\prime} - 6r\delta^2\psi^2 P_1^\circ - 4r\varphi^2 P_1^\circ + 3r\varphi\psi P_1^{\circ\prime}$$

is negative. Since $P_1^\circ > 0$, it suffices to state that the quadratic form in variables φ, ψ

$$6\delta^2\psi^2 + 4\varphi^2 + (16\delta^2 - 3)\frac{P_1^{\circ\prime}}{P_1^{\circ}}\varphi\psi$$

is positive definite. We see that the discriminant $D(\delta^2)$ of this form has two positive distinct roots and, therefore, we can always choose the required δ. The theorem is proved.

The proof of the analogous assertion for the resonance $\lambda_1 + \lambda_2 + \lambda_3 = 0$ is somewhat more difficult.

Theorem 4. If system (1.1) possesses the (third-order) resonance $\lambda_1 + \lambda_2 + \lambda_3 = 0$, then the Liapunov-instability of the equilibrium position follows from its instability in second order.

Proof. In analogy with the preceding, the question of the stability of the equilibrium position of system (1.1) is reduced to the investigation of the following system:

$$\frac{d\rho_\alpha^2}{dt} = 2\rho_1\rho_2\rho_3 P_\alpha(\bar\psi) + \rho_\alpha^2 (C_1^\alpha\rho_1^2 + C_2^\alpha\rho_2^2 + C_3^\alpha\rho_3^2 + S^\alpha) + R_{\alpha-1}, \quad (2.7)$$
$$\alpha = 1, 2, 3$$
$$\frac{d\rho_\beta^2}{dt} = \rho_\beta^2 (C_1^\beta\rho_1^2 + C_2^\beta\rho_2^2 + C_3^\beta\rho_3^2 + S^\beta) + R_{\beta-1},$$
$$\beta = 4, \ldots, l$$
$$\frac{d\bar\psi}{dt} = \rho_1\rho_2\rho_3\left(\frac{P_1'}{\rho_1^2} + \frac{P_2'}{\rho_2^2} + \frac{P_3'}{\rho_3^2}\right) + L_1\rho_1^2 + L_2\rho_2^2 + L_3\rho_3^2 + N^1 + R_l$$
$$P_\alpha(\bar\psi) = A_\alpha \cos\bar\psi + B_\alpha \sin\bar\psi, \qquad \bar\psi = \varphi_1 + \varphi_2 + \varphi_3,$$
$$R_i = R_i(\rho_1, \ldots, \rho_l, \bar\psi, \varphi_j)$$
$$N^1 = \sum_{j=4}^{l} L_j\rho_j^2, \qquad P_\alpha' = \frac{dP_\alpha}{d\bar\psi}, \qquad S^\alpha = \sum_{j=4}^{l} C_j^\alpha\rho_j^2$$

A_α, B_α, C_i^j, L_j are real coefficients, R_i denotes terms of higher order in comparison with the computed ones. The necessary and sufficient conditions for the existence of the invariant ray

$$\rho_\alpha = k_\alpha b(t), \qquad k_\alpha > 0, \qquad \alpha = 1, 2, 3$$
$$db^2/dt = \varkappa^2 b^3, \qquad \varkappa \neq 0, \quad \bar\psi = \psi_0 = \text{const}$$

are the following:

$$k_2^2 P_1(\psi_0) = k_1^2 P_2(\psi_0), \qquad k_3^2 P_1(\psi_0) = k_1^2 P_3(\psi_0) \quad (2.8)$$
$$\frac{P_1'(\psi_0)}{k_1^2} + \frac{P_2'(\psi_0)}{k_2^2} + \frac{P_3'(\psi_0)}{k_3^2} = 0, \qquad P_\alpha(\psi_0) > 0, \quad \alpha = 1, 2, 3$$

Using these conditions we write system (2.7) "in a neighborhood" of the invariant ray. For this we introduce the new coordinates r, φ, θ, ψ

$$\rho_1 = k_1 r \cos(\theta + \theta_0)\cos(\varphi + \pi/4)$$
$$\rho_2 = k_2 r \cos(\theta + \theta_0)\sin(\varphi + \pi/4)$$
$$\rho_3 = k_3 r \sin(\theta + \theta_0), \qquad \psi = \bar\psi - \psi_0$$
$$(\cos\theta_0 = \sqrt{2}/\sqrt{3}, \qquad \sin\theta_0 = 1/\sqrt{3})$$

The values $\varphi = 0$, $\theta = 0$, $\psi = 0$ correspond to the invariant ray. Expanding the right-hand sides of the tranformed system in Taylor series in a nighborhood of $\varphi = \theta = \psi = 0$, by taking (2.8) into account we obtain

$$\frac{dr}{dt} = \frac{r^2}{\sqrt{3}} P_3^\circ k_3^{12} + M^\circ r^3 + \frac{F^\circ}{2} r + Q_0 \tag{2.9}$$

$$\frac{d\varphi}{dt} = \frac{r}{2\sqrt{3}} (-4P_3^\circ k_3^{12}\varphi + B\psi) + Kr^2 + \frac{1}{4}(S^2 - S^1) + Q_1$$

$$\frac{d\theta}{dt} = \frac{r}{\sqrt{6}} (P_3^{\circ\prime} k_3^{12}\psi - 2\sqrt{2} P_3^\circ k_3^{12}\theta) + \frac{1}{2\sqrt{2}} N_3 r^2 + \frac{1}{2\sqrt{2}} S^3 - \frac{F^\circ}{2\sqrt{2}} + Q_2$$

$$\frac{d\psi}{dt} = \frac{r}{\sqrt{3}} (-3P_3^\circ k_3^{12}\psi - 2B\varphi - 3\sqrt{2} P_3^{\circ\prime} k_3^{12}\theta) + L^\circ r^2 + N^1 + Q_3$$

$$\frac{d\rho_\alpha^2}{dt} = \rho_\alpha^2 (N_\alpha r^2 + S^\alpha) + Q_\alpha, \qquad \alpha = 4, \ldots, l$$

$$B = P_2^{\circ\prime} k_2^{13} - P_1^{\circ\prime} k_1^{23}, \qquad F_j = \frac{1}{3}(C_j^1 + C_j^2 + C_j^3),$$

$$F^\circ = \sum_{j=4}^{l} F_j \rho_j^2, \qquad k_\alpha^{\beta\gamma} = \frac{k_\beta k_\gamma}{k_\alpha}$$

$$K = -{}^1/_2 (M_1 k_1^2 + M_2 k_2^2 + M_3 k_3^2), \quad L^\circ = {}^1/_3 (L_1 k_1^2 + L_2 k_2^2 + L_3 k_3^2$$

$$M_j = C_j^1 - C_j^2, \quad M^\circ = \frac{1}{6}(F_1 k_1^2 + F_2 k_2^2 + F_3 k_3^2), \quad N^1 = \sum_{j=4}^{l} L_j \rho_j^2$$

$$N_\alpha = \frac{1}{3}(C_1^\alpha k_1^2 + C_2^\alpha k_2^2 + C_3^\alpha k_3^2)$$

$$P_\alpha^\circ = P_\alpha(\psi_0), \quad P_\alpha^{\circ\prime} = P_\alpha'(\psi_0), \quad S^\alpha = \sum_{j=4}^{l} C_j^\alpha \rho_j^2$$

($Q_0, \ldots, Q_l$ denote higher-order terms). By analogy with the preceding we can verify that for a suitable choice of δ and for sufficiently small r

$$F = 4\varphi^2 + \psi^2 + \delta^2\theta^2 + \rho_4 + \ldots + \rho_l - r$$

is a Chetaev function for system (2.9).

3. Interaction of resonances. Definition. Two resonances with the resonant vectors $k_1 (k_{11}, \ldots, k_{1l})$ and $k_2 (k_{21}, \ldots, k_{2l})$ are said to be independent if the resonance relations do not have common frequencies, i. e. if

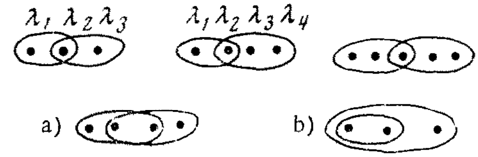
$$\sum_{j=1}^{l} |k_{1j}| \, |k_{2j}| = 0$$

A system having independent resonances splits up (in the second order) into noninterconnected subsystems (in suitable coordinates) ; therefore, the stability or instability in second order of the equilibrium position depends on whether all the resonances are unessential or at least one of them is essential.

We say that s resonances are linked in m frequencies (eigenvalues) if m frequencies occur in the resonance relations considered. For the (third-order) resonances we are studying the cases $m = 1, 2, \; s = 2$ (m cannot equal three since zero frequencies are absent). It is convenient to use the following linkage schemes:

A $m=1,\ s=2$

B $m=2,\ s=2$



A. We first examine the case $m = 1,\ s = 2$. Let us show that if both resonances are unessential, then the equilibrium position of the system being considered is stable. Let us verify the validity of this statement by the example of interaction of the following resonances: $\lambda_2 - 2\lambda_1 = 0,\ \lambda_4 - \lambda_2 - \lambda_3 = 0$. The remaining cases are analyzed analogously. Under the conditions being considered the truncated system (1.1) in polar coordinates has the following form: (we write down only the equations for ρ_α)

$$d\rho_1^2 / dt = 2\rho_1^2\rho_2 P_1(\psi_1), \quad d\rho_2^2 / dt = 2\rho_1^2\rho_2 P_2(\psi_1) + 2\rho_2\rho_3\rho_4 Q_1(\psi_2) \tag{3.1}$$

$$d\rho_3^2 / dt = 2\rho_2\rho_3\rho_4 Q_2(\psi_2),$$

$$d\rho_4^2 / dt = 2\rho_2\rho_3\rho_4 Q_3(\psi_2) \quad d\rho_\alpha^2 / dt = 0, \quad \alpha = 5, \ldots, l$$

$$P_j = A_j \cos\psi_1 + B_j \sin\psi_1, \quad j = 1,\ 2, \quad \psi_1 = \varphi_2 - 2\varphi_1$$

$$Q_k = C_k \cos\psi_2 + D_k \sin\psi_2, \quad k = 1,\ 2,\ 3, \quad \psi_2 = \varphi_4 - \varphi_2 - \varphi_3$$

Here $A_j,\ B_j,\ C_k,\ D_k$ are real coefficients. By hypothesis, $P_2 = -k^2 P_1$ (see (1.8)), while the determinants

$$\mathrm{D}_1 = \begin{vmatrix} C_2 & D_2 \\ C_3 & D_3 \end{vmatrix}, \quad \mathrm{D}_2 = \begin{vmatrix} C_3 & D_3 \\ C_1 & D_1 \end{vmatrix}, \quad \mathrm{D}_3 = \begin{vmatrix} C_1 & D_1 \\ C_2 & D_2 \end{vmatrix}$$

have the same sign (see [2]) ; to be specific let $D_i > 0$. We can verify that system (3.1) then has the integral

$$I = \mathrm{D}_1 k^2 \rho_1^2 + \mathrm{D}_1 \rho_2^2 + \mathrm{D}_2 \rho_3^2 + \mathrm{D}_4 \rho_4^2 + \sum_{j=5}^{l} \rho_j^2$$

whose existence guarantees stability.

Now suppose that at least one of the resonances is essential. We shall show that in this case the equilibrium position is unstable. Let us consider the interaction of two resonances of type 1 : 1 : 1

$$\lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_1 + \lambda_4 + \lambda_5 = 0$$

of which the first is essential. By this example it is easy to ascertain the general course of the proof for any two third-order resonances. Under the given conditions the normal form of system (1.1) is (the equations for the conjugate quantities are analogous):

$$y_1^\cdot = \lambda_1 y_1 + B_1 y_2^* y_3^* + B_2 y_4^* y_5^*$$

$$y_2^\cdot = \lambda_2 y_2 + B_3 y_1^* y_3^*, \quad y_3^\cdot = \lambda_3 y_3 + B_4 y_1^* y_2^*$$

$$y_4^\cdot = \lambda_4 y_4 + B_5 y_1^* y_5^*, \quad y_5^\cdot = \lambda_5 y_5 + B_6 y_1^* y_4^*$$

$$y_\alpha^\cdot = \lambda_\alpha y_\alpha, \quad \alpha = 6, \ldots, l$$

We show that this system has a growing solution. Having set $y_4 = \ldots = y_l = 0$, we obtain a system which by hypothesis possesses an invariant ray. Thus, the equilibrium of system (1.1) having two third-order resonances linked in one frequency is stable in second-order if both resonances are unessential, and is unstable if at least one of them is essential. This last statement is valid, obviously, for any number of resonances. Thus the situation corresponds completely to the already examined case of independent resonance.

B. Let the resonances be linked in two frequencies. This case differs qualitatively from the ones preceding in that the interaction of two unessential resonances can lead to instability. Let us consider a system depending on a parameter β and let us show that the equilibrium position is stable for $\beta > \beta_1$ and is unstable for $\beta < \beta_2$

Let $\lambda_2 + 2\lambda_1 = 0$, $\lambda_1 - \lambda_2 + \lambda_3 = 0$, let both resonances be unessential, and let the system in polar coordinates have the form

$$d\rho_1^2 / dt = -{}^5/_2\rho_1^2\rho_2 \sin \psi_1 + 2\rho_1\rho_2\rho_3 \sin \psi_2$$

$$d\rho_2^2 / dt = 10\rho_1^2\rho_2 \sin \psi_1 - 2\beta\rho_1\rho_2\rho_3 \sin \psi_2$$

$$d\rho_3^2 / dt = 3\rho_1\rho_2\rho_3 \sin \psi_2$$

$$\frac{d\psi_1}{dt} = 5\rho_1^2\rho_2\left(-\frac{1}{2\rho_1^2} + \frac{1}{\rho_2^2}\right)\cos\psi_1 - \rho_1\rho_2\rho_3\left(\frac{2}{\rho_1^2} + \frac{\beta}{\rho_2^2}\right)\cos\psi_2$$

$$\frac{d\psi_2}{dt} = 5\rho_1^2\rho_2\left(\frac{1}{4\rho_1^2} + \frac{1}{\rho_2^2}\right)\cos\psi_1 + \rho_1\rho_2\rho_3\left(\frac{1}{\rho_1^2} - \frac{\beta}{\rho_2^2} + \frac{3}{2\rho_3^2}\right)\cos\psi_2$$

This system has the integral $I = 4\rho_1^2 + \rho_2^2 + {}^2/_3(\beta - 4)\rho_3^2$, therefore, the equilibrium position is stable for $\beta > 4$. We see that for $\beta < {}^5/_2$ the system's solution is the following invariant ray:

$$\rho_1 = b(t), \quad \rho_2 = 2\sqrt{\frac{5-2\beta}{3}}\, b(t), \quad \rho_3 = 2b(t)$$

$$b^\cdot(t) > 0, \quad b(0) > 0, \quad \psi_1 = \psi_2 = \pi / 2$$

so that the equilibrium position is unstable. The Chetaev function is easily written down.

An analogous example can be cited for the case B (a): $\lambda_1 + \lambda_2 - \lambda_3 = 0$, $\lambda_2 + \lambda_3 - \lambda_4 = 0$ and

$$\frac{d\rho_1^2}{dt} = \rho_1\rho_2\rho_3 \sin\psi_1, \quad \frac{d\rho_2^2}{dt} = -2\rho_1\rho_2\rho_3 \sin\psi_1 + 6\rho_2\rho_3\rho_4 \sin\psi_2$$

$$\frac{d\rho_3^2}{dt} = 2\rho_1\rho_2\rho_3 \sin\psi_1 - 3\rho_2\rho_3\rho_4 \sin\psi_2, \quad \frac{d\rho_4^2}{dt} = \rho_2\rho_3\rho_4 \sin\psi_2$$

Only terms of the form $f(\rho)\cos\psi_\alpha$ occur in the equations for $\psi_1^\cdot$ and $\psi_2^\cdot$. For $\beta > 12$ this system has a positive-definite integral $I = \rho_1^2 + \rho_2^2 + \rho_3^2 / 2 + (\beta / 2 - 6)\rho_4^2$, while for $\beta < 2$ it has the invariant ray

$$\rho_1 = \rho_4 = b(t), \quad \rho_2 = 2b(t), \quad \rho_3 = \sqrt{2-\beta}\, b(t)$$

$$b(0) > 0, \quad b^\cdot(t) > 0, \quad \psi_1 = \psi_2 = \pi / 2$$

The author thanks the director and participants of V.V.Rumiantsev's seminar for useful discussions.

REFERENCES

1. Khazin, L.G., On the stability of Hamiltonian systems in the presence of resonances. PMM Vol.35, № 3, 1971.

2. Ibragimova, N.K., On the stability of certain systems in the presence of a resonance (English translation). Pergamon Press, J. USSR Comput. Mat. mat. Phys. Vol.6, № 5, 1966.

Translated by N.H.C.